Problem. There are five heads and fourteen legs in a family. How many people and how many dogs are in the family?

Solution. Let x be the number of people and y be the number of dogs.

Want: Value of x and y that satisfy the hypothesis: 5 heads and 14 legs in total.

Example: If $x = 2$ and $y = 3$ then

Heads : $2 + 3 = 5$

Legs : $2 \cdot 2 + 3 \cdot 4 = 4 + 12 = 16$

This does not work

Insted of trial and error we will use algebra to solve

Since there are x people and y dogs, and each person and dog has one head, the total number of heads is $x + y$. But, there must be 5 heads; so $x + y = 5$ — (i)

Since each person has two legs and each dog has four legs, the total number of legs is $2x + 4y$. But, there must be 14 legs; so $2x + 4y = 14$ —— (ii). By (i) and (ii),

$$x + y = 5 \quad (i)$$
$$2x + 4y = 14 \quad (ii)$$

Multiplying equation (i) by (-2) we get

$$-2x - 2y = -10 \quad (iii)$$

Adding equation (ii) and (iii) we get,

$$4y - 2y = 14 - 10$$
$$\Rightarrow 2y = 4$$
$$\Rightarrow y = \frac{4}{2}$$
$$\Rightarrow y = 2.$$

Substituting $y = 2$ into equation (i) we get
$$x + 2 = 5$$
$$\Rightarrow x = 3.$$

Therefore there are 3 people and 2 dogs.

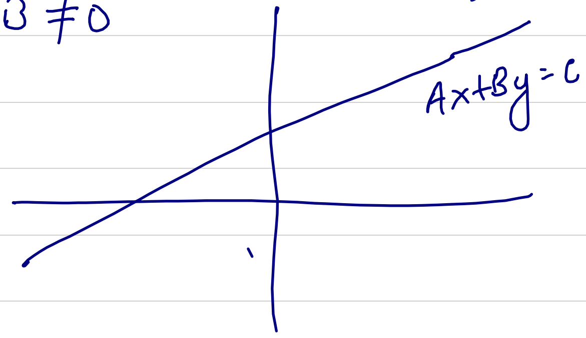
System of 2 linear equations in 2 variables

An equation in variables x and y is said to be linear if it is of the form:
$$Ax + By = C$$
where A, B and C are constants.

Note that in Cartesian coordinates the equation $Ax + By = C$ represents a line.

why?
$$Ax + By = C$$
$$\Rightarrow By = -Ax + C$$
$$\Rightarrow y = -\frac{A}{B}x + \frac{C}{B}$$

This is the line with slope $\left(-\frac{A}{B}\right)$ and y-intercept $\frac{C}{B}$ provided $B \neq 0$



A system of 2 linear equations is given by

$$A_1 x + B_1 y = C_1 \qquad \text{(i)}$$
$$A_2 x + B_2 y = C_2 \qquad \text{(ii)}$$

where $A_1, B_1, A_2, B_2, C_1, C_2$ are constants.

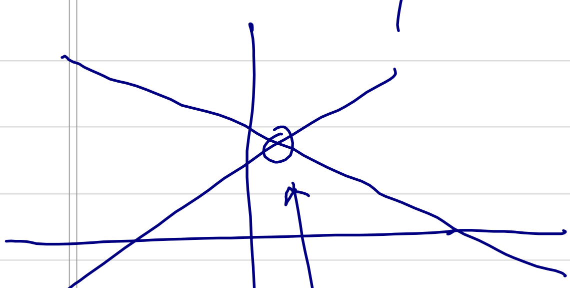
Solving the system means finding the values of x and y that satisfy both equations simultaneously.

Ques. Can a system of two linear equations have more than one solution?

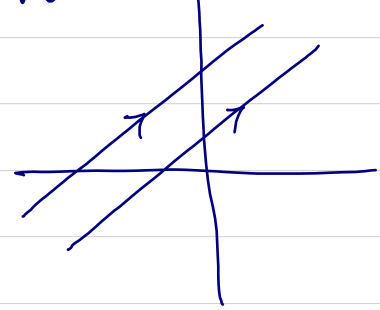
Ans. This can be answered algebraically, but there is a more intuitive, geometric answer:

$A_1 x + B_1 y = C_1$ represents a line in the plane and so does $A_2 x + B_2 y = C_2$. Since we want those values of x and y that satisfy both equations simultaneously, such a value lies in the intersection of the two lines.

What are the possible configurations?

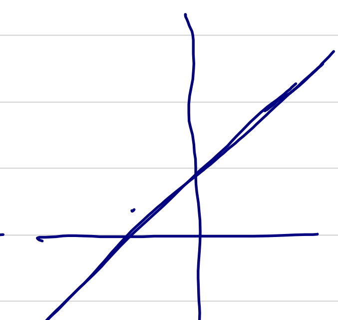


One solution



no solution

Do not intersect



every point on the line is a solution; there are infinitely many solutions when the two lines are the same.

So the answer is yes.

Substitution Method:

Solve
$$x + 2y = 6 \qquad \text{(i)}$$
$$3x - y = 11 \qquad \text{(ii)}$$

Solution. Idea: Solve the first or second equation for either x or y. Then substitute that value into the other equation.

From equation (i), subtracting $2y$ on both sides,
$$x = 6 - 2y$$

Now substituting this value of x into equation (ii) we get,
$$3(6-2y) - y = 11$$
or, $18 - 6y - y = 11$

or, $18 - 7y = 11$

or, $18 - 11 = 7y$

or, $7y = 7$

or, $y = 1.$

Substituting this value of y into equation (i) (Note it is equally correct to substitute into equation (ii)) we get
$$x + 2(1) = 6$$
or, $x = 6 - 2$

or, $x = 4$

Therefore, $x = 4$ and $y = 1.$

Exercises:

Solve:

(a)
$$2x + y = 3$$
$$4x + 2y = 4$$

(b)
$$x + 2y = 1$$
$$2x + 4y = 2.$$

Elimination method

Solve

$$-4x + 3y = 23 \qquad (i)$$
$$12x + 5y = 1 \qquad (ii)$$

Solution. Idea: Eliminate one of the variables (x or y). Multiply one or both equations by constants such that when you add the resulting equations, one of the variable cancels out.

I have decided to eliminate x. If I get $-12x$ in the first equation I can eliminate x.

Multiplying equation (i) by 3 we get

$$-12x + 9y = 69 \qquad (iii)$$

Adding equation (iii) and (ii) we have

$$\begin{array}{r} -12x + 9y = 69 \\ 12x + 5y = 1 \\ \hline 0 + 14y = 70 \end{array}$$

or, $14y = 70$

or, $y = \dfrac{70}{14} = 5$

Substituting this value of y into equation (ii) we get

$$12x + 5(5) = 1$$

or, $12x + 25 = 1$

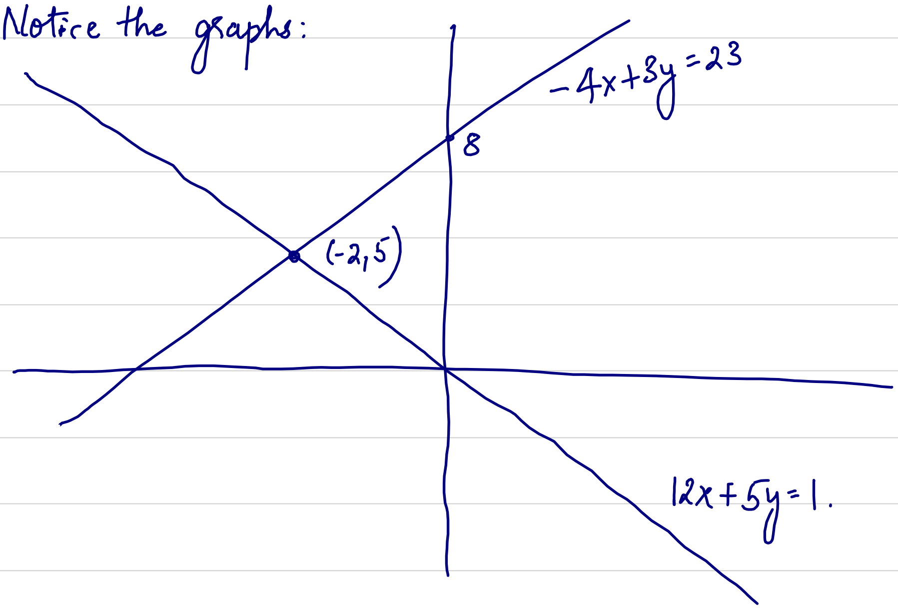
or, $12x = -24$

or, $x = \frac{-24}{12}$

or, $x = -2$.

Therefore, $x = -2$ and $y = 5$. $\square$

Notice the graphs:



Exercise

Solve

$$3x + 2y = 1$$
$$5x + 7y = 9$$

Sometimes there are no solutions:

Solve
$$-x+y=7 \qquad (i)$$
$$2x-2y=4 \qquad (ii)$$

Solution. Multiplying equation (i) by 2 we have
$$-2x+2y=14 \qquad (iii)$$

Adding equations (i) and (iii) we get
$$\begin{array}{r} -2x+2y=14 \\ 2x-2y=4 \\ \hline 0=20 \end{array}$$

This is a contradiction. This system is said to be <u>inconsistent</u>.

Notice the graphs:

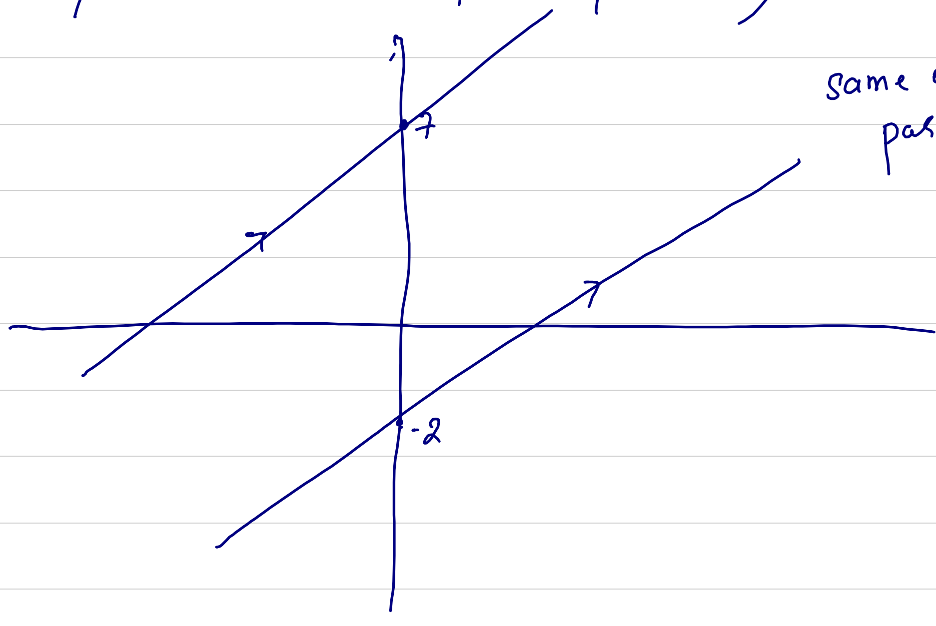
$-x+y=7$

$y=x+7$ (slope 1, y-int 7)

$2x-2y=4$

$2y=2x-4$

$y=x-2$ (slope 1, y-int -2)



same slope
parallel lines

Sometimes there are infinitely many solutions:

Solve
$$7x + y = 2 \qquad (i)$$
$$-14x - 2y = -4 \qquad (ii)$$

Soln. Multiplying equation (i) by 2 we get
$$14x + 2y = 4 \qquad (iii)$$

Adding equations (ii) and (iii) we have
$$\begin{array}{r} 14x + 2y = 4 \\ -14x - 2y = -4 \\ \hline 0 = 0 \end{array}$$

This is true for any x and y on the two lines. In other words, the two equations represent the same line. So every point on the line is a solution.

The solution is the line $y = -7x + 2$. There are infinitely many solutions. In this case the system is said to be <u>dependent</u>.

<u>Exercises</u>

Solve (a)
$$x - 5y = 2$$
$$-10x + 50y = -20$$

(b)
$$x - y = 14$$
$$-x + y = 9.$$

Graphing method

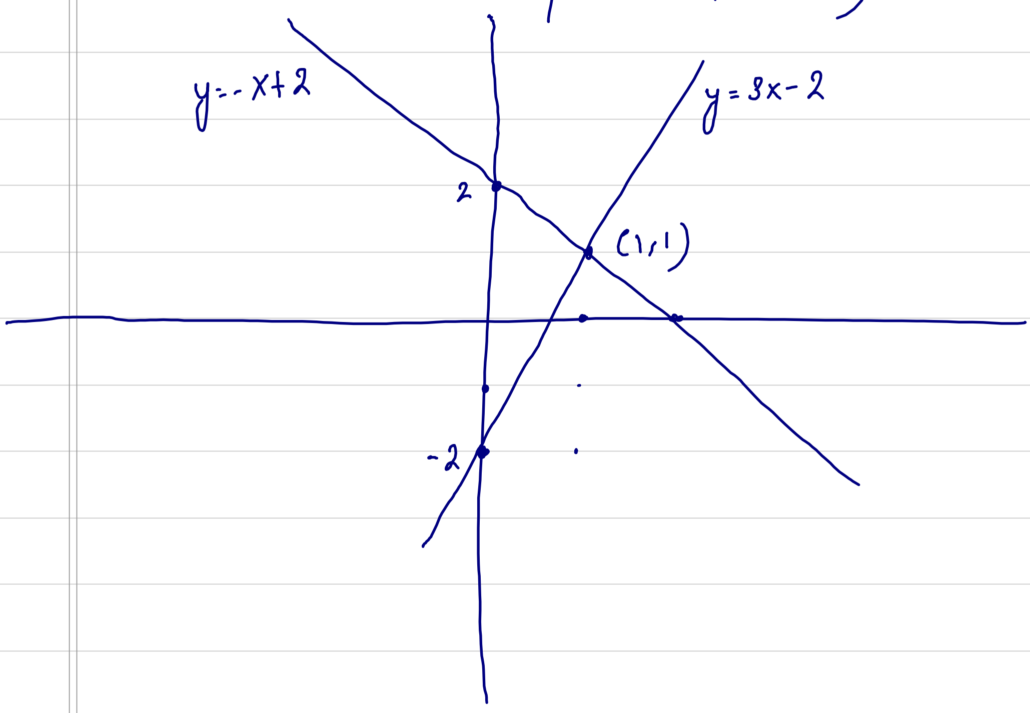
Solve

$$x + y = 2 \qquad (i)$$
$$3x - y = 2. \qquad (ii)$$

Solution Idea: Graph the two lines. Find the point of intersection. If no intersection, conclude that there is no solution. If they are the same line, conclude the line is the solution.

From eq. (i), $y = -x + 2$
(Slope -1, Y-int. 2)

From eq. (ii), $y = 3x - 2$
(Slope 3, Y-int -2)



Thus, $x = 1$ and $y = 1$ is the unique solution.